

# Superfluidity in Super-Yang-Mills Theory

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The AdS/CFT correspondence suggests that there is a point in the phase diagram of strongly interacting gauge-theory matter where the viscosity approaches zero. This paper analyses the possibility that this point represents a superfluid and that the system near this point in the phase diagram can be described by a Landau fluid. Superfluid vortices are constructed and the AdS analogue of vorticity quantisation is described. The production of vortices in the quark-gluon plasma during heavy ion collisions is discussed.

## I. INTRODUCTION

Experiments with heavy ion collisions at RHIC [1] have shown that the quark gluon plasma close to deconfinement can be successfully modelled using relativistic hydrodynamics. The properties of the quark gluon plasma should be described by quantum chromodynamics, but this theory is notoriously difficult to apply in the strong coupling regime. The AdS/CFT correspondence, developed from superstring theory, gives a much simpler theoretical framework for certain strongly coupled gauge theories [2, 3, 4]. The prime example is the  $\mathcal{N} = 4$  superconformal Yang-Mills theory. In the AdS/CFT correspondence, the thermal properties of the gauge-theory matter are related to black hole thermodynamics in 5 dimensions [4, 5, 6, 7, 8, 9].

Although  $\mathcal{N} = 4$  superconformal Yang-Mills theory can only be a toy model of quantum chromodynamics, it has been suggested that it might share some of the features of real quark physics close to the deconfinement phase transition [10, 11, 12, 13]. One influential result which arose in the context of the AdS/CFT correspondence was the ratio of shear viscosity  $\eta$  to entropy density  $s$  in superconformal  $\mathcal{N} = 4$  Yang-Mills theory, [14],

$$\frac{\eta}{s} = \frac{1}{4\pi} \quad (1)$$

in natural units ( $\hbar = c = k = 1$ ). The results from heavy ion collisions suggest that the viscosity of the quark gluon plasma might be close to the AdS/CFT prediction [15, 16].

We shall be interested here in what happens to gauge theory matter in the limit  $s \rightarrow 0$  with non-zero number density, when the viscosity vanishes. The known mechanism for vanishing viscosity is superfluidity, which is due to the Bose-Einstein condensation of a composite operator [17]. We shall consider the possibility that this is what happens here, i.e. that as  $s \rightarrow 0$  the system becomes a superfluid and can be described by a condensate.

When the phase of the system lies close to  $s = 0$ , then only part of the system would be in the condensate and the remainder would be in thermal excitations. In the superfluid context, this type of fluid is called a Landau fluid. Under the AdS/CFT correspondence, we would expect that two different black holes states should correspond to a single set of quark gas phase parameters. This is exactly the situation which one often finds when dealing with black holes in Anti de Sitter space (e.g. [18]). The smaller black hole is usually discarded, but we now look on this hole as representing one of the two components of the binary fluid.

In order to fix our ideas, consider the thermodynamics of a charged black hole in Anti-de Sitter space. The black hole corresponds to an equilibrium thermal field theory on a 3-sphere of radius  $l$  [4]. The electric charge of the black hole is related to a conserved quantity in the thermal system which is similar to baryon number [5] and the thermodynamic state can be described by a chemical potential  $\mu$  and the temperature  $T$ . There is a phase transition in the black hole thermodynamics [18] which corresponds to a quark deconfinement transition [19].

The phase diagram is shown in figure 1, where the deconfined phase has been split into two regions  $Ia$  and  $Ib$  with  $\mu < \mu_c$  and  $\mu \geq \mu_c$  respectively. There are two black hole solutions for each point in region  $Ia$ , and one black hole solution for each point in region  $Ib$ . If  $\mu \rightarrow \mu_c$  in region  $Ia$ , the black hole area of the smaller hole approaches zero, and according to our hypothesis it represents the superfluid component of a Landau fluid. The superfluid fraction is suppressed by its Gibbs free energy. At the end of the phase transition line lies a point  $(T, \mu) = (0, \mu_c)$  where  $s \rightarrow 0$  and the fluid would become a pure superfluid.

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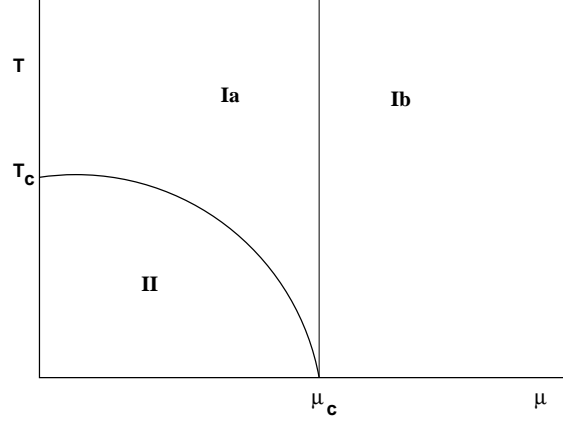


FIG. 1: The  $(T, \mu)$  phase diagram for Anti de Sitter space. Two charged black hole solutions exist for parameters in region *Ia*, and one charged black hole solution for parameters in region *Ib*. Region *I* is separated from region *II* by a second order phase transition. For supersymmetric black holes,  $\mu_c = \sqrt{3}$  in our units where the gravitational coupling strength  $\kappa_5 = 1$ .

The effective equation for a stationary condensate  $\Psi$  takes the form of a relativistic Gross-Pitaevskii equation (e.g. see [20, 21]). For simplicity we shall assume local interaction terms,

$$-\nabla^2 \Psi + g^2 |\Psi|^2 \Psi + m_B^2 \Psi = \mu^2 \Psi, \quad (2)$$

for some the constants  $g$  and  $m_B$ . The conserved charge  $N_B$  is given by

$$N_B = 2\mu \int \Psi^* \Psi d\Omega_3, \quad (3)$$

where  $d\Omega_3$  is the volume element on  $S_3$ .

The AdS/CFT correspondence can lead to some useful information about the couplings in the Gross-Pitaevskii equation. Consider the ground state with wave function  $\Psi_0 = (\mu^2 - m_B^2)^{1/2}/g$ . When combined with the formula for the conserved charge (3), this gives an upper bound on the coupling constant  $g^2 < 4\pi^2 \mu^3 l^3 / N_B$ . If the quark theory has gauge group  $SU(N)$ , then the radius of the anti de Sitter space is determined by the formula  $4\pi^2 l^3 = N^2$  [2]. We can also identify  $\mu$  with the value  $\mu_c$  obtained through black hole thermodynamics. The upper bound on the coupling becomes

$$g^2 \leq N^2 \mu_c^3 / N_B. \quad (4)$$

Both large and small coupling regimes are possible depending on the rank of the gauge group and the conserved charge.

Now consider what happens when the fluid rotates. The corresponding charged rotating black hole solutions were found by Chong et al [22]. The black holes rotate, but the geometry of the  $S_3$  on which the fluid lives remains unaffected by the rotation [9]. There are two independent axes of rotation on  $S_3$  which lie on two disconnected circles at  $\theta = 0$  and  $\theta = \pi$  in the Euler angle parameterisation  $(\theta, \psi, \phi)$ . The  $S_3$  metric in these coordinates is,

$$ds^2 = \frac{l^2}{4} (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{l^2}{4} (d\psi^2 + \cos \theta d\phi^2). \quad (5)$$

A simple class of vortex solutions to the Gross-Pitaevskii equation can be constructed by taking an ansatz

$$\Psi = R(\theta) e^{in_a(\phi+\psi)/2 + in_B(\phi-\psi)/2} \quad (6)$$

where  $n_a$  and  $n_B$  are integers. This reduces the Gross-Pitaevskii equation to an ordinary differential equation in  $z = -\cos \theta$ ,

$$(1 - z^2) \frac{d^2 R}{dz^2} - 2z \frac{dR}{dz} - \frac{n_a^2}{2(1 - z)} R - \frac{n_b^2}{2(1 + z)} R + \nu(\nu + 1) R - \frac{g^2 l^2}{4} R^3 = 0. \quad (7)$$

where  $\nu(\nu + 1) = (\mu^2 - m_b^2)l^2/4$ . The boundary conditions are  $R = 0$  at  $z = -1$  if  $n_B \neq 0$  and  $R = 0$  at  $z = 1$  if  $n_a \neq 0$ .

The vortex solutions represent one or two disconnected vortices at  $z = \pm 1$ . The two-vortex solutions only exist when  $\nu > 1$ . The vortices are thin compared to the radius of the three sphere  $l$  for large values of  $N$ . At more moderate values of  $N$ , the vortices have a size comparable to the three sphere and the modulus of the wave function is always smaller than the ground state value.

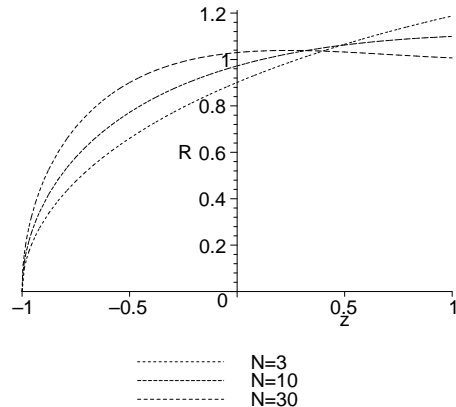


FIG. 2: The scaled wave function  $gR/\mu$  for a single vortex centred at  $\theta = 0$  or  $z = -1$ . The thickness of the vortex depends on the rank of the gauge group  $N$  but not on the conserved charge  $n_B$ . The mass  $m_B = 0$ .

Solutions of the Gross-Pitaevskii equation can also be described in terms of relativistic fluid dynamics [20]. The fluid-flux covector  $(n_B, n_B \mathbf{u})$  is related to the wave function by

$$n_B = 2\mu\Psi^*\Psi, \quad n_B \mathbf{u} = -i(\Psi^* d\Psi - \Psi d\Psi^*). \quad (8)$$

The fluid property of most interest in the case of vortex solutions is the circulation around curves  $\Gamma$ , defined by

$$C = \int_{\Gamma} \mathbf{u}. \quad (9)$$

For the solution ansatz (6), the circulation is constant outside the vortex cores, and given by  $C_a = 2\pi n_a/\mu$  or  $C_b = 2\pi n_b/\mu$  for curves around one or the other axis of rotation. In the non-relativistic limit,  $\mu \approx m_B$  and these reduce to the familiar quantisation of circulation.

The quantised quantities are really the angular momenta of the vortex solutions,

$$J_a = n_a \text{ and } J_b = n_b. \quad (10)$$

These carry across to the black hole solutions related to the gauge theory matter through the AdS/CFT correspondence. Quantum gravity is involved because the angular momenta of the classical black hole solutions are not quantised. In fact, only the  $N \rightarrow \infty$  limit corresponds to classical gravity, and in this limit the Gross-Pitaevskii equation breaks down as the thickness of the vortex solutions tends to zero.

There is strong evidence to support the quantisation of black hole angular momentum. AdS black holes in 3 dimensions, for example, can be described by a conformal algebra which implies states of quantised mass and angular momentum [23]. These play an important role in the statistical approach to black hole entropy for near-extremal rotating black holes in 5 dimensions [24]. The idea here is that the mass, angular momentum and area can be discussed entirely in terms of the geometry close to the horizon, which takes the limiting form of a black hole in 3 dimensions. Again, there is a conformal algebra which implies that the angular momenta along the two axes of rotation are quantised.

The above ideas are closely related to approaches to black hole entropy which are based on string duality. These relate the properties of the black hole to an ensemble of  $D$ -brane states [25]. When applied to extremal rotating black holes in 5 dimensions, one finds that the black hole angular momenta are given by quantised  $D$ -brane charges [26].

The points in the phase diagram which are associated with zero viscosity are dense, low temperature states. This seems to be related more to neutron stars than to heavy ion collisions. The situation when transformed to Minkowski space is, in fact, rather more interesting. There is a conformal transformation  $f$  which maps  $S_3 \times R$  to Minkowski space  $R^4$  which takes the time coordinate  $t$  and the azimuthal polar coordinate  $\chi$  to the Minkowski light cone coordinates  $u$  and  $v$ ,

$$u = l \tan \left( \frac{t}{2l} - \frac{\chi}{2} \right) \quad (11)$$

$$v = l \tan \left( \frac{t}{2l} + \frac{\chi}{2} \right). \quad (12)$$

This transformation reduces to stereographic projection at  $t = 0$ , but in general surfaces of constant time  $t$  do not map to surfaces of constant Minkowski time. The timelike killing vector is mapped to

$$f_* \partial_t = \frac{1}{2} (P + K) \quad (13)$$

where  $P$  and  $K$  are the generators of time translation and timelike conformal transformations respectively [27]. The conformal symmetry can be used to relate operator traces, but thermal states on  $S_3$  are related to non-thermal and non-stationary states in Minkowski space.

Consider operators  $\varphi$  and  $\varphi'$  with conformal weight 1. The conformal symmetry relates ensemble averages on  $S_3 \times R$  to ensemble averages on Minkowski space,

$$\text{tr}(\rho \varphi(x) \varphi(x')) = \Omega(x) \Omega(x') \text{tr}(\rho' f^* \varphi'(x) f^* \varphi'(x')), \quad (14)$$

where  $\Omega$  is the conformal factor. The thermal states with  $\rho = e^{-\beta H}$  are related to ensemble averages in Minkowski space with a density matrix

$$\rho' = e^{-\beta(H' + K')/2} \quad (15)$$

where  $H'$  and  $K'$  are the Minkowski space Hamiltonian and generator of conformal transformations respectively. The new density matrix does not commute with the Hamiltonian, or with any other Poincaré group generator.

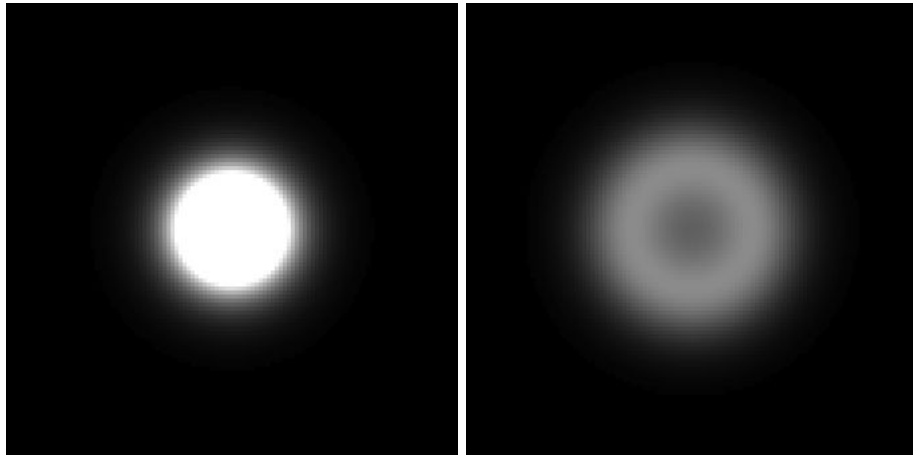


FIG. 3: The white sphere shows the spatial distribution of the energy density of the gauge matter fluid at the initial time  $t = 0$  when it is at rest and at a later time  $t = l$  as it expands.

The conformal symmetry allows the energy density of the gauge-theory matter in Minkowski space to be expressed in terms of the conformal factor  $\Omega$  and its derivatives. The result is shown in figure 3. The Minkowski space ensemble represents a ball of fluid which starts from rest and then expands and dissipates away.

Even though the Minkowski system is non-thermal, it is possible to define a local temperature by examining the KMS condition of periodicity in imaginary time when the points  $x$  and  $x'$  lie close to the centre of the ball of fluid. The effective inverse temperature  $\beta'$  at the centre is given in an elementary way by substituting  $t = i\beta$  in eq. (11),

$$\beta' = l \tanh\left(\frac{\beta}{2l}\right) \quad (16)$$

The  $T \rightarrow 0$  limit which we associated with the superfluid state corresponds to a temperature  $T' \rightarrow l^{-1}$  at the centre of the ball of fluid. The image of the Landau fluid in Minkowski space is therefore effectively at or above the temperature which we associated with a deconfinement phase transition.

The vortex solutions can also be mapped to flat space by the conformal transformation (11) and (12). The images of the solutions are circular ring vortices or line vortices. Double vortex solutions are mapped to interlinked vortices which lie in two orthogonal planes. Figure 4 shows a section through a ring vortex and a line vortex at the initial Minkowski time.

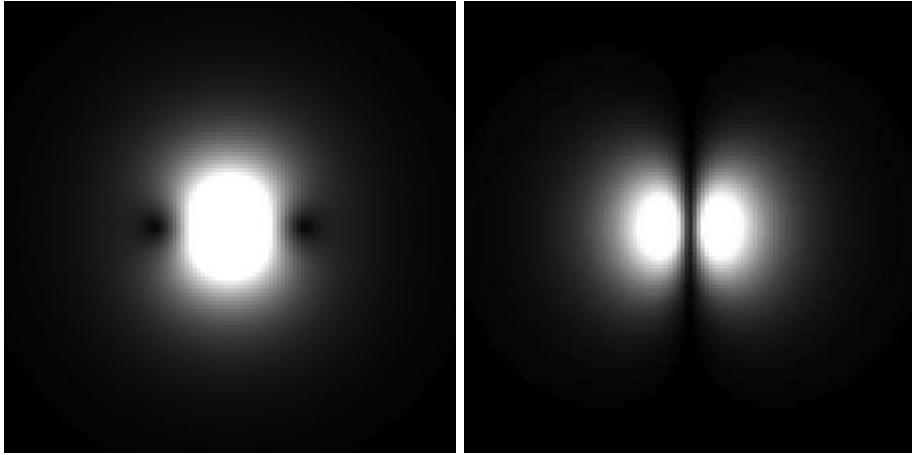


FIG. 4: The spatial distribution of the number density of the gauge matter fluid is shown on a vertical slice through a horizontal ring vortex (left) and a vertical line vortex (right). The fluid is ejected from the vortex core. These figures correspond to the middle curve in figure 2.

We turn finally to the question of whether these superfluid concepts can tell us anything about the quark-gluon plasma. In particular, can we produce superfluid vortices in heavy ion collisions? In relativistic heavy ion collisions, Lorentz contraction of the nuclei results in a planar collision geometry. After a short time has elapsed, thermal behaviour sets in and the energy becomes spread over an increasingly spherical region [28]. The baryon chemical potential in ultrarelativistic collisions [29, 30, 31, 32]  $\mu \approx 10\text{MeV}$ , appears to be far smaller compared to the temperature of the quark-gluon plasma  $T \approx 170\text{MeV}$  than we would need for a Landau fluid description.

The picture of gauge matter obtained from spherical black holes differs from those models in which gauge-theory matter is modelled by an evolving black brane [10, 11, 12]. These are based on planar geometry, but it would be interesting to extend these ideas to base the evolution on spherical background geometry using conformal mappings of the type discussed here.

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